1 LASSO with Splitting Schemes

We now consider the LASSO problem:

$$F(\boldsymbol{\theta}) = \frac{1}{2}\|A\boldsymbol{\theta} - \mathbf{y}\|^2 + \lambda\|\boldsymbol{\theta}\|_1 \ \to \ \min_{\boldsymbol{\theta}\in\mathbb{R}^d}, \tag{1}$$

where $A \in \mathbb{R}^{n\times d}$, $\mathbf{y} \in \mathbb{R}^n$, and $\lambda > 0$. Let $L = \|A^\top A\|$, $\boldsymbol{\theta}^*$ a minimizer, and $0 < h \leq 1/L$.

Theorem 1 (LT zero algorithmic bias)**.** *Let* $\bar{\boldsymbol{\theta}}_K = \frac{1}{K}\sum_{k=0}^{K-1}\boldsymbol{\theta}_k$*. Then*

$$F(\bar{\boldsymbol{\theta}}_K) - F(\boldsymbol{\theta}^*) \ \leq \ \frac{\|\boldsymbol{\theta}_0 - \boldsymbol{\theta}^*\|^2}{2hK}. \tag{2}$$

Theorem 2 (Strang-ISTA convergence with bias floor)**.**

$$F(\bar{\boldsymbol{\theta}}_K) - F(\boldsymbol{\theta}^*) \ \leq \ \frac{\|\boldsymbol{\theta}_0 - \boldsymbol{\theta}^*\|^2}{2hK} + C_D \cdot h^2 \cdot \lambda^2. \tag{3}$$

Remark 1 (Constant C_D)**.** *Numerical estimation yields* $C_D \approx 0.77$*. Analytical conjecture:*

$$C_D = \frac{\|A \cdot \mathrm{sign}(\boldsymbol{\theta}^*)\|^2}{8}. \tag{4}$$

The Strang fixed point satisfies $\bar{\boldsymbol{\theta}}^h - \boldsymbol{\theta}^* \approx \frac{h\lambda}{2}\,\mathrm{sign}(\boldsymbol{\theta}^*)$.

Corollary 1 (Splitting order matters)**.** *LT/ISTA: zero bias. Strang-ISTA: bias* $C_D h^2\lambda^2 + O(h^3)$.